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THE GENERAL PROPERTIES OF THE EQUATIONS OF THE NON-LINEAR THEORY OF ELASTICITY FOR PIECEWISE-LINEAR POTENTIALS†

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A theory of elasticity for piecewise-linear potentials is constructed assuming that the elastic potential consists of two terms, one of which depends on the hydrostatic pressure and other on the equivalent stress Σ, which is a homogeneous function of the first power of the stress deviator. These assumptions limit the class of possible models compared with the previous assumptions [1], but they are more practical since, when choosing a certain expression for Σ to determining the model, two experiments on uniaxial and volume extension-contraction are sufficient. The use of piecewise-linear expressions for Σ in some cases introduces certain simplifications, and some new properties of the models arise which do not occur for smooth convex functions of Σ. Thus, under certain conditions it becomes possible for stress and strain surfaces of discontinuity to exist, characteristic surfaces occur, and problems arise regarding the uniqueness of the solution. The solution of these problems is considered in this paper. Copyright © 1996 Elsevier Science Ltd.

The possibility of constructing a theory of elasticity for piecewise-linear potentials was suggested by Ivlev in [2]. The possibility of constructing a theory of elasticity with a constant Poisson's ratio, in which, for uniaxial extension there is a linear relationship between the stresses and strains, was considered in [1].

Note that the use of piecewise-linear plasticity conditions has led to some progress in the theory of plasticity [3–5].

1. Small deformations of an elastic medium are related to the stresses as follows:

$$e_{ij} = \partial U/\partial\sigma_{ij} \tag{1.1}$$

We will consider isotropic media, in which the potential U depends only on the invariants of the stress tensor, while the change in the volume is uniquely defined by the value of the first invariant of the stress tensor $\sigma = 1/3(\sigma_1 + \sigma_2 + \sigma_3)$, where σ_i are the principal stresses. This condition will be satisfied if we postulate the following expression for the potential

$$U(\sigma_{ij}) = U_1(\sigma) + U_2(\Sigma) \tag{1.2}$$

where Σ is a homogeneous function of the first degree of the components of the stress deviator, i.e. $\Sigma(t(\sigma_1 - \sigma), t(\sigma_2 - \sigma), t(\sigma_3 - \sigma)) = t\Sigma(\sigma_1 - \sigma, \sigma_2 - \sigma, \sigma_3 - \sigma)$. The surface Σ = const in the space of the principal stresses $\sigma_1, \sigma_2, \sigma_3$ is represented by a cylinder with generatrices parallel to the straight line $\sigma_1 = \sigma_2 = \sigma_3$. This surface will be defined if the section of the cylinder by the deviator plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$ is specified. Without loss of generality we can assume that, for a uniaxial stresses $\sigma_1 \neq 0$, $\sigma_2 = \sigma_3 = 0$, we have the equation

$$\Sigma\left(\frac{2}{3}\sigma_1,\ -\frac{1}{3}\sigma_2,\ -\frac{1}{3}\sigma_3\right) = \sigma_1$$

We will confine ourselves to considering normally isotropic media, i.e. such that a change in the sign of the stresses leads to a similar change in the sign of the strain. These assumptions lead to limitations on the choice of the functions $U_1(\sigma)$ and $U_2(\Sigma)$, which must be even functions of their arguments.

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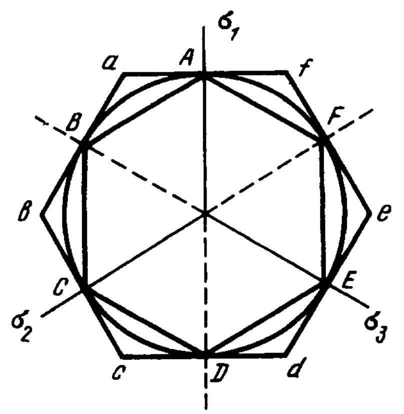

Fig. 1.

In Fig. 1 we show a section of the cylinder Σ = const by the deviator plane, where the equation of the inner hexagon is

$$\Sigma = \max|\sigma_i - \sigma_j| = \text{const} \tag{1.3}$$

and the equation of the outer hexagon is

$$\Sigma = \max|\sigma_i - \sigma| = \text{const} \tag{1.4}$$

These hexagons are noteworthy in that all possible convex surfaces Σ = const are situated between them. The circle shown in Fig. 1 corresponds to the classical theory of elasticity. The equation of this circle has the form

$$\Sigma = \sqrt{\frac{3}{2}}((\sigma_1 - \sigma)^2 + (\sigma_2 - \sigma)^2 + (\sigma_3 - \sigma)^2)^{1/2} = \text{const} \tag{1.5}$$

If we postulate a linear relationship between the stresses and strains for omnidirectional and uniaxial stresses

$$e = \frac{1}{3}(e_1 + e_2 + e_3) = \frac{1-\nu}{E}\sigma, \quad e_1 = \frac{\sigma_1}{E} \tag{1.6}$$

the choice of expression (1.5) for Σ leads to a linear equation of the theory of elasticity. At the same time, the use of expressions (1.3) or (1.4) for Σ is of undoubted interest as an extremal version of the theory.

Substituting (1.2) into (1.1) we obtain

$$e_{ij} = \frac{1}{3}U_1'(\sigma)\delta_{ij} + U_2'(\Sigma)\frac{\partial\Sigma}{\partial\sigma_{ij}} \tag{1.7}$$

We have the following relations for the derivatives of the isotropic function Σ

$$\frac{\partial\Sigma}{\partial\sigma_{ij}} = \frac{\partial\Sigma}{\partial\sigma_1}l_il_j + \frac{\partial\Sigma}{\partial\sigma_2}m_im_j + \frac{\partial\Sigma}{\partial\sigma_3}n_in_j \tag{1.8}$$

where l_i, m_i, n_i are the direction cosines of the principal axes of the stresses tensor, which are related by the condition

$$l_il_j + m_im_j + n_in_j = \delta_{ij} \tag{1.9}$$

If the surface Σ = const is smooth, no difficulties arise in calculating the derivatives from Eqs (1.8).

If the stresses on the surface Σ = const correspond to a corner point, the derivative must be taken in the generalized sense. Suppose the stresses correspond to the intersection of the smooth surfaces $\Sigma = \Sigma_1$ and $\Sigma = \Sigma_2$. Then, the normal to the surface degenerates into a fan of normals, i.e.

$$\frac{\partial \Sigma}{\partial \sigma_i} = \alpha \frac{\partial \Sigma_1}{\partial \sigma_i} + \beta \frac{\partial \Sigma_2}{\partial \sigma_i}, \quad \alpha \geqslant 0, \quad \beta \geqslant 0 \tag{1.10}$$

Since, by definition, Σ, Σ_1, Σ_2 are homogeneous functions of the first degree, then by Euler's theorem for homogeneous functions we have

$$\frac{\partial \Sigma}{\partial \sigma_i}\sigma_i = \Sigma, \quad \frac{\partial \Sigma_1}{\partial \sigma_i}\sigma_i = \Sigma_1, \quad \frac{\partial \Sigma_2}{\partial \sigma_i}\sigma_i = \Sigma_2 \tag{1.11}$$

Convoluting (1.10) with σ_i and taking (1.11) into account, we obtain $\Sigma = \alpha\Sigma + \beta\Sigma_2$. Since on the surface considered $\Sigma = \Sigma_1 = \Sigma_2$, we have $\alpha + \beta = 1$, and relations (1.10) take the form

$$\frac{\partial \Sigma}{\partial \sigma_i} = \alpha \frac{\partial \Sigma_1}{\partial \sigma_i} + (1-\alpha) \frac{\partial \Sigma_2}{\partial \sigma_i}, \quad 0 \leqslant \alpha \leqslant 1 \tag{1.12}$$

The derivatives to the surface (1.3) at the corner points and the conditions for which the stresses correspond to the corner points, are shown in Table 1. Similar data are given in Table 2 for surface (1.4).

The modes corresponding to the corner points, and the modes corresponding to the smooth parts of the potential, will hold in certain regions of the strain, the boundaries of which are unknown in advance. For each mode we obtain a closed system of equations, if we combine with relations (1.7)–(1.9)

Table 1

Mode	Constraints	$\frac{\partial \Sigma}{\partial \sigma_1}$	$\frac{\partial \Sigma}{\partial \sigma_2}$	$\frac{\partial \Sigma}{\partial \sigma_3}$
A	$\sigma_1 > \sigma_2 = \sigma_3$	1	$-\alpha$	$-1+\alpha$
B	$\sigma_2 < \sigma_1 = \sigma_3$	α	-1	$1-\alpha$
C	$\sigma_3 > \sigma_1 = \sigma_2$	$-\alpha$	$-1+\alpha$	1
D	$\sigma_1 < \sigma_2 = \sigma_3$	-1	α	$1-\alpha$
E	$\sigma_2 > \sigma_1 = \sigma_3$	$-\alpha$	1	$-1+\alpha$
F	$\sigma_3 < \sigma_1 = \sigma_2$	α	$1-\alpha$	-1

Table 2

Mode	Constraints	$\frac{\partial \Sigma}{\partial \sigma_1}$	$\frac{\partial \Sigma}{\partial \sigma_2}$	$\frac{\partial \Sigma}{\partial \sigma_3}$
a	$\sigma_1 - \sigma = -\sigma_3 + \sigma > 0$	$\frac{1}{2}+\frac{1}{2}\alpha$	$\frac{1}{2}-\alpha$	$-1+\frac{1}{2}\alpha$
b	$\sigma_1 - \sigma = -\sigma_2 + \sigma > 0$	$\frac{1}{2}+\frac{1}{2}\alpha$	$-1+\frac{1}{2}\alpha$	$\frac{1}{2}-\alpha$
c	$\sigma_3 - \sigma = -\sigma_2 + \sigma > 0$	$\frac{1}{2}-\alpha$	$-1+\frac{1}{2}\alpha$	$\frac{1}{2}+\frac{1}{2}\alpha$
d	$\sigma_3 - \sigma = -\sigma_1 + \sigma > 0$	$-1+\frac{1}{2}\alpha$	$\frac{1}{2}-\alpha$	$\frac{1}{2}+\frac{1}{2}\alpha$
e	$\sigma_2 - \sigma = -\sigma_1 + \sigma > 0$			
f	$\sigma_2 - \sigma = -\sigma_3 + \sigma > 0$			

the equations relating the components of the stress tensor with the principal values, the expression for the components of the strains in terms of the displacements, and the equilibrium equations

$$\sigma_{ij} = \sigma_1 l_i l_j + \sigma_2 m_i m_j + \sigma_3 n_i n_j, \quad e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) \tag{1.13}$$

$$\sigma_{ij,j} + F_i = 0 \tag{1.14}$$

Here u_i are the displacements and F_i are the volume forces. Note that at the corner points of the surface Σ the principal stresses are related linearly, which reduces the number of statistical functions required by one, but a new unknown function α arises as a compensation in kinematics, which has to be determined when solving the boundary-value problems. The modes on the arcs hold if the constraints given in the tables hold, and $0 \leqslant \alpha \leqslant 1$. Modes corresponding to the faces will occur if completely defined constraints are imposed on the principal stresses.

For the hexagon Σ, chosen in the form (1.3), the following relations must be satisfied

$$\begin{aligned}
&AB: \Sigma = \sigma_1 - \sigma_2, \quad \sigma_1 \geqslant \sigma_2 \geqslant \sigma_3\\
&BC: \Sigma = \sigma_3 - \sigma_2, \quad \sigma_3 \geqslant \sigma_1 \geqslant \sigma_2\\
&CD: \Sigma = \sigma_3 - \sigma_1, \quad \sigma_3 \geqslant \sigma_2 \geqslant \sigma_1\\
&DE: \Sigma = \sigma_2 - \sigma_1, \quad \sigma_2 \geqslant \sigma_3 \geqslant \sigma_1\\
&EF: \Sigma = \sigma_2 - \sigma_3, \quad \sigma_2 \geqslant \sigma_1 \geqslant \sigma_3\\
&FA: \Sigma = \sigma_1 - \sigma_3, \quad \sigma_1 \geqslant \sigma_2 \geqslant \sigma_3
\end{aligned}$$

When Σ is chosen in the form (1.4) the following relations must be satisfied

$$\begin{aligned}
&ab: \Sigma = \frac{3}{2}(\sigma_1 - \sigma), \quad -\sigma_3 + \sigma \leqslant \sigma_1 - \sigma, \quad -\sigma_2 + \sigma \leqslant \sigma_1 - \sigma\\
&bc: \Sigma = \frac{3}{2}(-\sigma_2 + \sigma), \quad -\sigma_2 + \sigma \geqslant \sigma_1 - \sigma, \quad -\sigma_2 + \sigma \geqslant \sigma_3 - \sigma\\
&cd: \Sigma = \frac{3}{2}(\sigma_3 - \sigma), \quad \sigma_3 - \sigma \geqslant -\sigma_1 + \sigma, \quad \sigma_3 - \sigma \geqslant -\sigma_2 + \sigma\\
&de: \Sigma = \frac{3}{2}(-\sigma_1 + \sigma), \quad -\sigma_1 + \sigma \geqslant \sigma_2 - \sigma, \quad -\sigma_1 + \sigma \geqslant \sigma_3 - \sigma\\
&ef: \Sigma = \frac{3}{2}(-\sigma_1 + \sigma), \quad -\sigma_1 + \sigma \geqslant \sigma_2 - \sigma, \quad -\sigma_1 + \sigma \geqslant \sigma_3 - \sigma\\
&fa: \Sigma = \frac{3}{2}(-\sigma_3 + \sigma), \quad -\sigma_3 + \sigma \geqslant \sigma_1 - \sigma, \quad -\sigma_3 + \sigma \geqslant \sigma_3 - \sigma
\end{aligned}$$

The use of expressions (1.3) and (1.4) when solving boundary-value problems of the non-linear theory of elasticity leads to additional difficulties because of the need to choose the modes. Simplifications can be made in cases when the choice of mode can be made in advance or guessed, and also when the directions of at least one of the principal stresses are known.

2. We will investigate the possibility that discontinuous solutions exist for piecewise-linear potentials.

Suppose there is a surface Ω in the region V, in which the displacements are continuous, while the stresses and strains have a discontinuity. Using the geometrical conditions of compatibility [6] for jumps in the strains, we obtain the relations

$$\begin{aligned}
&[e_{ij}] = \tfrac{1}{2}(\bar{u}_i^{(1)} \nu_j + \bar{u}_j^{(1)} \nu_i),\\
&\bar{u}_i^{(1)} = du_i^+ / dn - du_i^- / dn
\end{aligned} \tag{2.1}$$

where u_i^+ and u_i^- are the displacements on different sides of the surface Ω and ν_i are the components of the vector normal to this surface.

The condition of continuity of the stress vector on Ω (the condition of equilibrium) has the form

$$[\sigma_{ij}]\nu_j = 0 \tag{2.2}$$

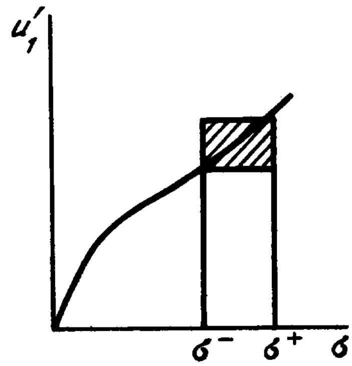

Fig. 2.

It follows from (2.1) and (2.2) that

$$(\sigma_{ij}^+ - \sigma_{ij}^-)e_{ij}^+ + (\sigma_{ij}^- - \sigma_{ij}^+)e_{ij}^- = [\sigma_{ij}][e_{ij}] = [\sigma_{ij}]\nu_j \bar{u}_i^{(1)} = 0 \tag{2.3}$$

Substituting the values of the strains, calculated from (1.7) into (2.3), we obtain after reduction

$$\begin{aligned}&(\sigma^+ - \sigma^-)(U_1'(\sigma^+) - U_1'(\sigma^-)) + (\Sigma^+ - \Sigma^-)(U_2'(\Sigma^+) - U_2'(\Sigma^-)) + \\ &+U_2'(\Sigma^+)(t^+\sigma_{ij}^+ - \sigma_{ij}^-)\frac{\partial\Sigma(\sigma_{ij}^+)}{\partial\sigma_{ij}^+} + U_2'(\Sigma^-)(t^-\sigma_{ij}^- - \sigma_{ij}^+)\frac{\partial\Sigma(\sigma_{ij}^-)}{\partial\sigma_{ij}^-} = 0\end{aligned} \tag{2.4}$$

Since $t^+\sigma_{ij}^+$ and σ_{ij}^- lie on one surface $\Sigma = \Sigma(\sigma_{ij})$, while $t^-\sigma_{ij}^-$ and σ_{ij}^+ lie on the surface $\Sigma = \Sigma(\sigma_{ij}^+)$, the conditions for the surface $\Sigma = \text{const}$ to be non-convex lead to the following inequalities

$$(t^+\sigma_{ij}^+ - \sigma_{ij}^-)\frac{\partial\Sigma(\sigma_{ij}^+)}{\partial\sigma_{ij}^+} \geqslant 0, \quad (t^-\sigma_{ij}^- - \sigma_{ij}^+)\frac{\partial\Sigma(\sigma_{ij}^-)}{\partial\sigma_{ij}^-} \geqslant 0 \tag{2.5}$$

The area of the square, shown hatched in Fig. 2, is always positive if $U_1'(\sigma)$ is a monotonically increasing function of σ. Hence it follows that

$$(U_1'(\sigma^+) - U_1'(\sigma^-))(\sigma^+ - \sigma^-) \geqslant 0 \tag{2.6}$$

where the equality in (2.6) is only obtained when $\sigma^+ = \sigma^-$ or $U_1'(\sigma^+) = U_1'(\sigma^-)$ (the latter only occurs for incompressible materials).

Similarly, we have for the monotonically increasing function $U_2'(\Sigma)$

$$(U_2'(\Sigma^+) - U_2'(\Sigma^-))(\Sigma^+ - \Sigma^-) \geqslant 0 \tag{2.7}$$

where the equality holds when $\Sigma^+ = \Sigma^-$ (or $U_2'(\Sigma^+) = U_2'(\Sigma^-)$ (the latter occurs for a reinforced material).

It follows from (2.4)–(2.7) that the following equalities hold on the surface of discontinuity

$$\Sigma(\sigma_{ij}^+) = \Sigma(\sigma_{ij}^-), \quad \sigma^+ = \sigma^- \tag{2.8}$$

$$(\sigma_{ij}^+ - \sigma_{ij}^-)\frac{\partial\Sigma(\sigma_{ij}^+)}{\partial\sigma_{ij}^+} = 0, \quad (\sigma_{ij}^- - \sigma_{ij}^+)\frac{\partial\Sigma(\sigma_{ij}^-)}{\partial\sigma_{ij}^-} = 0 \tag{2.9}$$

The condition for the surface $\Sigma = \text{const}$ to be convex has the form $(\sigma_{ij}^+ - \sigma_{ij}^-)\partial\Sigma(\sigma_{ij}^+)/\partial\sigma_{ij}^+ \geqslant 0$, and hence the values of σ_{ij}^- for fixed σ_{ij}^+ are such that the function

$$B = (\sigma_{ij}^+ - \sigma_{ij}^-)\frac{\partial\Sigma(\sigma_{ij}^+)}{\partial\sigma_{ij}^+} = \Sigma(\sigma_{ij}^+) - \sigma_{ij}^-\frac{\partial\Sigma(\sigma_{ij}^+)}{\partial\sigma_{ij}^+} = \Sigma(\sigma_{ij}^-) - \sigma_{ij}^-\frac{\partial\Sigma(\sigma_{ij}^+)}{\partial\sigma_{ij}^+} \tag{2.10}$$

has the minimum value in them.

The minimum of the function B, with conditions (2.8), occurs if

$$\frac{\partial\Sigma(\sigma_{ij}^-)}{\partial\sigma_{ij}^-}-\frac{\partial\Sigma(\sigma_{ij}^+)}{\partial\sigma_{ij}^+}+\psi_1\frac{\partial\Sigma(\sigma_{ij}^-)}{\partial\sigma_{ij}^-}+\psi_2\delta_{ij}=0 \tag{2.11}$$

Convoluting (2.11) with δ_{ij} we obtain $\psi_2 = 0$, convoluting (2.11) with σ_{ij}^- and taking (2.9) into account we obtain $\psi_1 = 0$, and from (2.11) we have

$$\frac{\partial\Sigma(\sigma_{ij}^-)}{\partial\sigma_{ij}^-}=\frac{\partial\Sigma(\sigma_{ij}^+)}{\partial\sigma_{ij}^+} \tag{2.12}$$

It follows from (2.12) that the principal directions of the tensors σ_{ij}^+ and σ_{ij}^- coincide on the surface of continuity and, consequently, Eqs (2.9) can be written in a system of coordinates coinciding with the principal directions of the stress tensor in the form

$$(\sigma_i^+-\sigma_i^-)\frac{\partial\Sigma(\sigma_i^+)}{\partial\sigma_i^-}=0,\quad (\sigma_i^--\sigma_i^+)\frac{\partial\Sigma(\sigma_i^-)}{\partial\sigma_i^-}=0 \tag{2.13}$$

Equation (2.13) can be satisfied if the surface Σ = const has plane sections, i.e. $\Sigma = a_i\sigma_i$, $a_1 + a_2 + a_3 = 0$, while σ_i^+ and σ_i^- lie in the same plane part. In this case, from the first relation of (1.13) and Eqs (2.2) and (2.8) we obtain the following relations for the jumps of the principal stresses

$$[\sigma_1]l_il_j\nu_j+[\sigma_2]m_im_j\nu_j+[\sigma_3]n_in_j\nu_j=0 \tag{2.14}$$

$$[\sigma_1]+[\sigma_2]+[\sigma_3]=0 \tag{2.15}$$

$$a_1[\sigma_1]+a_2[\sigma_2]+a_3[\sigma_3]=0 \tag{2.16}$$

It follows from (2.14) that

$$[\sigma_1]l_j\nu_j=0,\quad [\sigma_2]m_j\nu_j=0,\quad [\sigma_3]n_j\nu_j=0 \tag{2.17}$$

We conclude from (2.15) that the jumps of the two principal stresses cannot simultaneously vanish. To fix our ideas we will put $[\sigma_2] \neq 0$, $[\sigma_3] \neq 0$; then $m_j\nu_j = 0$ and $n_j\nu_j = 0$. Hence $l_j\nu_j = 1$ and $[\sigma_1] = 0$. It then follows from (2.15) and (2.16) that $a_2 = a_3 = a_{1/2}$. Hence, for compressible isotropic media ($U_1'(\sigma) \neq 0$), discontinuities of the stresses are only possible when the effective stress Σ is chosen in the form of the maximum reduced stress $\Sigma = \max|\sigma_i - \sigma|$. Here the direction cosines of the principal stresses are continuous, the effective principal stress (σ_1) is continuous, the normal to the surface of discontinuity coincides with the direction of the effective main direction ($l_j\nu_j = 1$), and the jumps in the other two principal stresses are equal in value but opposite in sign.

If the material is incompressible, we have $U_1'(\sigma) = 0$, and the condition (2.6) is satisfied for any σ^+ and σ^-, while the jumps in the stresses satisfy conditions (2.16) and (2.17).

Suppose $[\sigma_1] = 0$, $[\sigma_2] \neq 0$, $[\sigma_3] \neq 0$. From (2.16) and (2.17) we then have $m_i\nu_i = n_i\nu_i = 0$, $l_i\nu_i \neq 0$, $a_2[\sigma_2] + a_3[\sigma_3] = 0$. Hence, in an incompressible medium having a piecewise-linear potential, surfaces of discontinuity of the stresses are possible on which one principal stress, directed along the normal to the surface of discontinuity, is continuous, while the principal stresses lying in the tangential plane undergo a discontinuity.

Suppose $[\sigma_1] = [\sigma_2] = 0$, $[\sigma_3] \neq 0$. From (2.17) we then have $n_i\nu_i = 0$ and it follows from (2.16) that $a_3 = 0$. Consequently, if $\Sigma = \sigma_1 - \sigma_2$, a surface of discontinuity of the stresses is possible in an incompressible medium in which the maximum and minimum principal stresses are continuous, while only the intermediate principal stress, which lies in a plane tangential to the surface of the discontinuity, undergoes a discontinuity.

In all cases, the strains on the surface of discontinuity of the stresses are continuous, and consequently, the stresses on the surface of the discontinuity of the strains are continuous.

If, on a certain surface S, the stresses correspond to a smooth part of the function Σ, the deformations are continuous on the surface S. Hence, on the surface of discontinuity of the deformations, the stresses correspond to the corner points of the function Σ. In this case, relations (1.7) take the form

$$e_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}) = \tfrac{1}{3}U_1'(\sigma)\delta_{ij} + U_2'(\Sigma)\left(\alpha\frac{\partial\Sigma_1}{\partial\sigma_{ij}} + (1-\alpha)\frac{\partial\Sigma_2}{\partial\sigma_{ij}}\right) \tag{2.18}$$

Using the geometrical conditions of compatibility and relations (1.8) we obtain from (2.18)

$$\tfrac{1}{2}(\bar{u}_i^{(1)}\nu_j + \bar{u}_j^{(1)}\nu_i) = [\alpha]U_2'(\Sigma)(A_1 l_i l_j + A_2 m_i m_j + A_3 n_i n_j) \tag{2.19}$$

$$A_i = \partial\Sigma_1 / \partial\sigma_i - \partial\Sigma_2 / \partial\sigma_i$$

Convoluting (2.19) with δ_{ij} we obtain $\bar{u}_i^{(1)}\nu_i = 0$. Then, after convolution with ν_i and (2.19) we obtain

$$\tfrac{1}{2}\bar{u}_i^{(1)} = [\alpha]U_2'(\Sigma)(A_1 l_i l_k \nu_k + A_2 m_i m_k \nu_k + A_3 n_i n_k \nu_k) \tag{2.20}$$

Substituting (2.20) into (2.19) we obtain that the following relations are satisfied on the surface of discontinuity of the strains

$$A_1(l_i l_j - (l_i\nu_j + l_j\nu_i)l_k\nu_k) + A_2(m_i m_j - (m_i\nu_j + m_j\nu_i)m_k\nu_k) +$$

$$+ A_3(n_i n_j - (n_i\nu_j + n_j\nu_i)n_k\nu_k) = 0 \tag{2.21}$$

Of the six relations (2.21) only three are independent, since after convolution with δ_{ij} and ν_j they are reduced to a single equation. We obtain independent relations (2.21) by projecting them onto $l_i l_j$, $m_i m_j$, $n_i n_j$. They have the form

$$\begin{aligned} &A_1(1 - 2(l_k\nu_k)^2) = 0 \\ &A_2(1 - 2(m_k\nu_k)^2) = 0 \\ &A_3(1 - 2(n_k\nu_k)^2) = 0 \end{aligned} \tag{2.22}$$

Taking into account the fact that $A_1 + A_2 + A_3 = 0$ and $(l_k\nu_k)^2 + (m_k\nu_k)^2 + (n_k\nu_k)^2 = 1$, we conclude that Eqs (2.22) can only be satisfied when

$$\begin{aligned} &A_1 = 0, \quad (m_k\nu_k)^2 = (n_k\nu_k)^2 = \tfrac{1}{2}, \quad l_k\nu_k = 0 \\ &A_2 = 0, \quad (l_k\nu_k)^2 = (n_k\nu_k)^2 = \tfrac{1}{2}, \quad m_k\nu_k = 0 \\ &A_3 = 0, \quad (l_k\nu_k)^2 = (m_k\nu_k)^2 = \tfrac{1}{2}, \quad n_k\nu_k = 0 \end{aligned}$$

Hence, surfaces of discontinuity of the strains in non-linearly elastic media are only possible for potentials which depend on $\Sigma = \max|\sigma_i - \sigma_j|$ and which coincide with the surfaces of maximum shear.

3. We will consider the surfaces of discontinuity of the derivatives of the stresses and strains. We will assume that the stresses, strains and first derivatives of the stresses are continuous on these surfaces, while the first derivatives of the stresses and the second derivatives of the strains can have discontinuities. The geometrical conditions of compatibility can then be written in the form

$$[\sigma_{ij,k}] = \bar{\sigma}_{ij}^{(1)}\nu_k, \quad [u_{i,jk}] = \bar{u}_i^{(2)}\nu_j\nu_k \tag{3.1}$$

$$\bar{\sigma}_{ij}^{(1)} = [d\sigma_{ij} / dn], \quad \bar{u}_i^{(2)} = [d^2u_i / dn^2] = [\partial^2 u_i / \partial x_j \partial x_k]\nu_k\nu_j$$

From (1.7) and (1.13) we obtain that the following relations are satisfied on the surface of discontinuity

$$\bar{\sigma}_{ij}^{(1)}\nu_j = 0 \tag{3.2}$$

$$\frac{1}{2}(\bar{u}_i^{(2)}\nu_j + \bar{u}_j^{(2)}\nu_i) = \frac{1}{3}U_1''(\sigma)\bar{\sigma}^{(1)}\delta_{ij} + \left(U_2''(\Sigma)\frac{\partial\Sigma}{\partial\sigma_{ij}}\frac{\partial\Sigma}{\partial\sigma_{kl}} + U_2''(\Sigma)\frac{\partial^2\Sigma}{\partial\sigma_{ij}\partial\sigma_{kl}}\right)\bar{\sigma}_{kl}^{(1)} \tag{3.3}$$

Convoluting (3.3) with $\bar{\sigma}_{ij}^{(1)}$ and taking (3.2) into account we obtain

$$\frac{1}{3}U_1''(\sigma)(\bar{\sigma}^{(1)})^2 + U_2''(\Sigma)\left(\frac{\partial\Sigma}{\partial\sigma_{ij}}\bar{\sigma}_{ij}^{(1)}\right)^2 + U_2'(\Sigma)\frac{\partial^2\Sigma}{\partial\sigma_{ij}\partial\sigma_{kl}}\bar{\sigma}_{kl}^{(1)}\bar{\sigma}_{ij}^{(1)} = 0 \tag{3.4}$$

Since $U_1''(\sigma) \geqslant 0$, $U_i^2(\Sigma) \geqslant 0$, $U_2''(\Sigma) \geqslant 0$, Eq. (3.4) is satisfied if

$$U_1''(\sigma)\bar{\sigma}^{(1)} = 0, \quad \frac{\partial\Sigma}{\partial\sigma_{ij}}\bar{\sigma}_{ij}^{(1)} = 0, \quad \frac{\partial^2\Sigma}{\partial\sigma_{ij}\partial\sigma_{kl}}\bar{\sigma}_{kl}^{(1)}\bar{\sigma}_{ij}^{(1)} = 0 \tag{3.5}$$

For convex surfaces Σ = const we have the inequality

$$\frac{\partial^2\Sigma}{\partial\sigma_{ij}\partial\sigma_{kl}}\bar{\sigma}_{kl}^{(1)}\bar{\sigma}_{ij}^{(1)} \geqslant 0 \tag{3.6}$$

where the equality and (3.6) only occurs when $\bar{\sigma}_{ij}^{(1)} = 0$.

Hence, if the surface Σ = const is convex, the relations of the non-linear theory of elasticity in the form (1.7) do not permit the existence of surfaces of discontinuity of the derivatives of the stresses and strains, i.e. $\bar{\sigma}_{ij}^{(1)} = 0$, $\bar{u}_i^{(2)} = 0$. It can be shown that when the condition of convexity is satisfied, discontinuities of the derivatives of the stresses and strains of any order are impossible, and the general equations of the non-linear theory of elasticity are elliptic.

The direction Δ_{ij} will be called [6] the direction in which the surface Σ is flattened, if the following equalities hold

$$\frac{\partial\Sigma}{\partial\sigma_{ij}}\Delta_{ij} = 0, \quad \frac{\partial^2\Sigma}{\partial\sigma_{ij}\partial\sigma_{kl}}\Delta_{ij}\Delta_{kl} = 0 \tag{3.7}$$

If a flattening direction occurs on the surface Σ, the equality in (3.6) can also be satisfied when $\bar{\sigma}_{ij}^{(1)} \neq 0$, but sometimes the tensor $\bar{\sigma}_{ij}^{(1)}$ may coincide with the flattening tensor. We know [6], that for isotropic media the principal axes of the flattening tensor coincide with the principal axes of the tensor σ_{ij}, and hence Eqs (3.7) can be represented in the form

$$\frac{\partial\Sigma}{\partial\sigma_i}\Delta_i = 0, \quad \frac{\partial^2\Sigma}{\partial\sigma_i\partial\sigma_j}\Delta_i\Delta_j = 0 \tag{3.8}$$

(Δ_i are the principal values of the tensor Δ_{ij}). The second condition of (3.8) is satisfied identically on the faces of the piecewise-linear surfaces Σ = const.

Taking (3.8) and (3.5) into account we obtain that relations (3.2) and (3.3) will be satisfied if

$$\bar{\sigma}_1^{(1)}l_il_k\nu_k + \bar{\sigma}_2^{(1)}m_im_k\nu_k + \bar{\sigma}_3^{(1)}n_in_k\nu_k = 0 \tag{3.9}$$

$$\bar{\sigma}_1^{(1)} + \bar{\sigma}_2^{(1)} + \bar{\sigma}_3^{(1)} = 0 \tag{3.10}$$

$$a_1\bar{\sigma}_1^{(1)} + a_2\bar{\sigma}_2^{(1)} + a_3\bar{\sigma}_3^{(1)} = 0 \tag{3.11}$$

Comparing (3.9)–(3.11) with relations (2.14)–(2.16), we obtain that the conclusions reached for the surfaces of discontinuity of the stresses can be reformulated without change for the surfaces of discontinuity of the derivatives of the stresses and strains.

If the stressed state corresponds to a singular point of the surface Σ = const, we obtain, on the surface of discontinuity of the derivatives of the stresses and strains,

$$\frac{1}{2}(\bar{u}_i^{(2)}\nu_j + \bar{u}_j^{(2)}) = \frac{1}{3}U_1''(\sigma)\bar{\sigma}^1\delta_{ij} + U_2'(\Sigma)\bar{\alpha}^{(1)}\left(\frac{\partial\Sigma_1}{\partial\sigma_{ij}} - \frac{\partial\Sigma_2}{\partial\sigma_{ij}}\right) +$$

$$+ U_2'(\Sigma)\left(\alpha\frac{\partial^2\Sigma_1}{\partial\sigma_{ij}\partial\sigma_{kl}} + (1-\alpha)\frac{\partial^2\Sigma_2}{\partial\sigma_{ij}\partial\sigma_{kl}}\right)\bar{\sigma}_{kl}^{(1)} + U_2''(\Sigma)\left(\alpha\frac{\partial\Sigma_1}{\partial\sigma_{kl}} + (1-\alpha)\frac{\partial\Sigma_2}{\partial\sigma_{kl}}\right)\times$$

$$\times\left(\alpha\frac{\partial\Sigma_1}{\partial\sigma_{ij}}+(1-\alpha)\frac{\partial\Sigma_2}{\partial\sigma_{ij}}\right)\bar{\sigma}_{kl}^{(1)} \tag{3.12}$$

Convoluting (3.12) with $\bar{\sigma}_{ij}^{(1)}$ and using (3.2) we have

$$\frac{1}{3}U_1''(\sigma)\bar{\sigma}^{(2)}+U_2'(\Sigma)\left(\alpha\frac{\partial^2\Sigma_1}{\partial\sigma_{ij}\partial\sigma_{kl}}+(1-\alpha)\frac{\partial^2\Sigma_2}{\partial\sigma_{ij}\partial\sigma_{kl}}\right)\bar{\sigma}_{ij}^{(1)}\bar{\sigma}_{kl}^{(1)}+$$

$$+U_2''(\Sigma)\bar{\alpha}^{(1)}\left(\frac{\partial\Sigma_1}{\partial\sigma_{ij}}-\frac{\partial\Sigma_2}{\partial\sigma_{ij}}\right)\partial\bar{\sigma}_{ij}^{(1)}+U_2''(\Sigma)\left(\left(\alpha\frac{\partial\Sigma_1}{\partial\sigma_{ij}}+(1-\alpha)\frac{\partial\Sigma_2}{\partial\sigma_{ij}}\right)\bar{\sigma}_{ij}^{(1)}\right)^2=0 \tag{3.13}$$

Assuming that the condition $\Sigma_1 = \Sigma_2$ holds on both sides of the surface of discontinuity, we obtain

$$\left(\frac{\partial\Sigma_1}{\partial\sigma_{ij}}-\frac{\partial\Sigma_2}{\partial\sigma_{ij}}\right)\bar{\sigma}_{ij}^{(1)}=0 \tag{3.14}$$

Taking into account the fact that $0 \leqslant \alpha \leqslant 1$, we obtain from (3.13) and (3.14)

$$U_1''(\sigma)\bar{\sigma}^{(1)}=0$$

$$\frac{\partial^2\Sigma_1}{\partial\sigma_{ij}\partial\sigma_{kl}}\bar{\sigma}_{ij}^{(1)}\bar{\sigma}_{kl}^{(1)}=\frac{\partial^2\Sigma_2}{\partial\sigma_{ij}\partial\sigma_{kl}}\bar{\sigma}_{ij}^{(1)}\bar{\sigma}_{kl}^{(1)}=\frac{\partial\Sigma_1}{\partial\sigma_{ij}}\bar{\sigma}_{ij}^{(1)}=\frac{\partial\Sigma_2}{\partial\sigma_{ij}}\bar{\sigma}_{ij}^{(2)}=0 \tag{3.15}$$

From the convexity condition we obtain that on the surface of discontinuity $\bar{\sigma}_{ij}^{(1)} = 0$.
Conditions (3.2) are then satisfied, and (3.12) can be written in the form

$$\tfrac{1}{2}(\bar{u}_i^{(2)}\nu_j+\bar{u}_j^{(2)}\nu_i)=U_2'(\Sigma)\bar{\alpha}^{(1)}(A_1l_il_j+A_2m_im_j+A_3n_in_j) \tag{3.16}$$

Comparing (3.16) and (2.20), we can conclude that the properties of the surfaces of discontinuity of the strain are identical with the properties of the surfaces on which the derivatives of the strains suffer a discontinuity.

Hence, the general relations of the non-linear theory of elasticity for potentials, chosen as functions Σ in the form (1.3) and (1.4), have characteristic surfaces.

When Σ is chosen in the form (1.4), this will represent surfaces orthogonal to the principal stress of maximum modulus, while the stressed state on the surface of discontinuity will correspond to the faces of the surface Σ = const.

4. The traditional consideration of the theory of uniqueness is based on the equation of virtual work, which we will write in the form

$$\int_V \sigma_{ij}e_{ij}dV=\int_\Omega \sigma_{ij}u_iN_jdS+\int_S(\sigma_{ij}^+-\sigma_{ij}^-)\nu_ju_idS+\int_V F_iu_idV \tag{4.1}$$

Here V is the volume of the body considered, Ω is the surface which bounds it, N_i is the normal to Ω, S is the surface of discontinuity of the stresses inside the volume V, σ_{ij}^+ and σ_{ij}^- are the stresses on the two sides of S, and ν_i is the normal to S outward from the "plus" region. Equation (4.1) is satisfied if u_i, e_i and σ_{ij} satisfy the second equation of (1.3) and Eq. (1.14).

For Eqs (1.7), (1.13) and (1.14) we will consider solutions which satisfy the boundary conditions on the surface Ω, namely

$$\sigma_{ij}N_j=p_i(x_k),\quad x_k\in\Omega_p;\quad u_i=f_i(x_k),\quad x_k\in\Omega_u \tag{4.2}$$

$$\varepsilon_{ijk}\sigma_{jm}N_mL_k=p_i(x_k),\quad u_iL_i=f_L(x_k),\quad x_k\in\Omega_{pu}$$

Here $p_i(x_k)$, $f_i(x_k)$, $f_L(x_k)$, $L_i(x_k)$ are specified functions on the corresponding surfaces, and $p_iL_i = 0$ on Ω_{pu}, $\Omega = \Omega_p + \Omega_u + \Omega_{pu}$. Conditions (2.2) are satisfied on the surfaces of discontinuity of the stresses.

Suppose we have two solutions $\sigma_{ij}^{(1)}$, $e_{ij}^{(1)}$, $u_i^{(1)}$ and $\sigma_{ij}^{(2)}$, $e_{ij}^{(2)}$, $u_i^{(2)}$ which satisfy the above conditions. Then, the difference between these two solutions, according to Eq. (4.1), must satisfy the equation

$$(\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)})(e_{ij}^{(1)} - e_{ij}^{(2)}) = 0 \tag{4.3}$$

From Eq. (4.3) we obtain the conclusions which follow from an analysis of (2.3). Hence it follows that for convex smooth surfaces Σ = const the distribution of the stresses and strains are uniquely defined. If the surface Σ = const has plane parts, the stresses $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ may differ in certain regions. Here $\Sigma(\sigma_{ij}^{(1)}) = \Sigma(\sigma_{ij}^{(2)})$, $\sigma^{(1)} = \sigma^{(2)}$ for compressible media, the principal axes of the tensors $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ coincide, the principal stresses $\sigma_{ij}^{(1)}$ and $\sigma_{ij}^{(2)}$ may differ but lie in the same plane part of the surface Σ = const, and the distribution of the strains in this case will be unique. If the surface Σ = const has corner points, the regions in which the stressed state corresponds to these points will be identical in both solutions, and in these regions the stresses will be identical. Consequently, for specified boundary conditions (4.2) the region V can be split uniquely into the sum of regions $V_k(V = \Sigma V_k)$ so that in each region the mode corresponding to the specific faces and edges of the surface Σ = const is satisfied.

Relations (1.7) were treated above as equations of the theory of elasticity. We can regard them simultaneously as relations of the deformation theory of plasticity in the case of an active load. Moreover, the use of the equivalent stress in the form (1.4) turns out to agree better with experimental data presented in [7], than the relations of the classical theory when the surfaces Σ are chosen in the form (1.5).

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REFERENCES

1. IVLEV D. D., The construction of a theory of elasticity. *Dokl. Akad. Nauk SSSR* **138**, 6, 1321–1324, 1961.
2. IVLEV D. D., The construction of the hydrodynamics of a viscous liquid. *Dokl. Akad. Nauk SSSR* **135**, 2, 280–282, 1960.
3. IVLEV D. D., *The Theory of Ideal Plasticity*. Nauka, Moscow, 1966.
4. ERKHOV M. I., *The Theory of Ideally Plastic Solids and Structures*. Nauka, Moscow, 1978.
5. HODGE F. G., *The Design of Structures Taking Plastic Deformations into Account*. Mashgiz, Moscow, 1963.
6. IVLEV D. D. and BYKOVTSEV G. I., *The Theory of a Reinforced Plastic Solid*. Nauka, Moscow, 1971.
7. ZHIGALKIN V. M., USOVA O. M. and SHEMYAKIN Ye. I., Simple and complex loading of steel at normal and low temperatures. *Proceedings of the 1st All-Union School-Seminar*, Leningrad, 1983.

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